XLVI. *Calculations in Spherical Trigonometry abridged. By* Iſrael Lyons. *In a Letter to Sir* John Pringle, *Bart. P. R. S.*

TO SIR JOHN PRINGLE, BART. P. R. S.

SIR,

Redde, July 6, 1775.

SINCE aſtronomical obſervations have been made with much greater preciſion than formerly, it became requiſite that the calculations correſponding to them ſhould likewiſe be made to much greater degrees of exactneſs. The ancient aſtronomers deſired only to make their obſervations and computations agree within a part of a degree; ſucceeding ones were ſatisfied when they correſponded within a minute; but no leſs exactneſs than ſeconds will content the moderns. The rules in ſpherical trigonometry being reduced to operations by logarithms, it is neceſſary to uſe ſuch a number of figures in the tables as will produce the required preciſion; this is very different in the various parts of the quadrant, inſomuch that if the arc is only one degree, four places of decimals in the logarithm of a ſine are ſufficient to determine the arc to which it belongs within a ſecond: whereas if the arc is 89°, there is a neceſſity

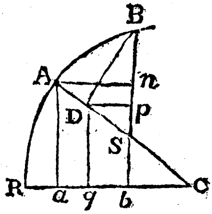
neceffity of ufing eight figures for the fame purpofe: thus, the logarithm fine of 89° 0′ 0″ is 9.9999338, the fame feven figures as for the logarithm fine of 89° 0′ 1″. From this confideration it follows, that the analogies commonly laid down and ufed for the folutions of fpherical triangles are not in all cafes equally convenient, and I might fay, equally accurate; and that it would be more eafy and exact in calculations to find what was required, by means of fines of arcs, which, being fmall, require the ufe of only a few places of figures. Now the cafes which often occur in aftronomy, where fpherical trigonometry can only be of ufe, are generally of fuch a nature that we know nearly, or at leaft within a few degrees, what the required fide or angle is, there is nothing therefore wanted but to find how much this quantity, or firft approximation, differs from the true value of the fide or angle. Thus in calculating the right afcenfion of any point of the ecliptic, whofe longitude and declination are known, inftead of finding the right afcenfion immediately, it will be more convenient to feek for the difference between the longitude and right afcenfion, which as it never exceeds $2\frac{1}{2}$°, four or five places of figures will always be fufficient to determine it within a fecond. And in other fimilar cafes, rules might be made agreeable to the exigency of each particular cafe, which would be better than the application of the general method of folution. Some examples of which fhall be fhewn in the

 following

following paper: the design of which is to point out a method of solving several of the most useful questions in spherical trigonometry in a manner somewhat similar to that used in approximating to the roots of algebraic equations. This method is founded on the following

LEMMA.

If the radius is supposed equal to unity, the sine of the sum of two arcs, α and β, is equal to sin. α + cos. $\alpha \times$ sin. β − sin. $\alpha \times$ vers. sin. β. And its cosine = cos. α − sin. $\alpha \times$ sin. β − cos. $\alpha \times$ vers. sin. β.



DEMONSTRATION.

Let the arc α be RA, and the arc β be AB, their sines Aa, BD, respectively; then Bb being drawn perpendicular to the radius CR will be the sine of $\alpha+\beta$. Draw Dp and An parallel to CR. Then, by similar triangles, CA : Ca :: BD : Bp, and CA : Aa :: AD : np. Therefore, Bb(=Aa+Bp−pn)=Aa+$\frac{Ca \times BD}{CA}$−$\frac{Aa \times AD}{CA}$; that is, sine $\overline{\alpha+\beta}$ = sin. α + cos. $\alpha \times$ sin. β − sin. $\alpha \times$ vers. sin. β.

In the same manner, drawing Dq parallel to Aa we may prove Cb(=Ca−bq−aq)=CA−$\frac{Aa \times BD}{CA}$−$\frac{Ca \times AD}{CA}$, or cos. $\overline{\alpha+\beta}$ = cos. α − sin. $\alpha \times$ sin. β − cos. $\alpha \times$ vers. β.

In

In what follows, for brevity ſake, the arc is expreſſed by a Greek letter; its ſine by the capital character; and the coſine by the ſmall italic character of the ſame letter. In this notation, the two theorems will ſtand thus, ſin. $\overline{\alpha+\beta} = \mathrm{A} + a\mathrm{B} - \mathrm{A} \times \text{vſ.}\, \beta$, and coſ. $\overline{\alpha+\beta} = a - \mathrm{AB} - a \times \text{vſ.}\, \beta$.

COROLLARY I.

Since the tangent is equal to the ſine divided by the coſine, we ſhall have

$$\text{Tang.}\ \overline{\alpha+\beta} = \frac{\mathrm{A} + a\mathrm{B} - \mathrm{A} \times \text{vſ.}\, \beta}{a - \mathrm{AB} - a \times \text{vſ.}\, \beta} = \frac{\mathrm{A}}{a} + \frac{\mathrm{B}}{a^2} + \frac{\mathrm{A}}{a^3} \times \text{vſ.}\, \beta \text{ nearly.}$$

COROLLARY II.

If we change the ſign of β, we ſhall have ſin. $\overline{\alpha-\beta} = \mathrm{A} - a\mathrm{B} - \mathrm{A} \times \text{vſ.}\, \beta$. Coſ. $\overline{\alpha-\beta} = a + \mathrm{AB} - a \times \text{vſ.}\, \beta$. And tang. $\overline{\alpha-\beta} = \frac{\mathrm{A}}{a} - \frac{\mathrm{B}}{a^2} + \frac{\mathrm{A}}{a^3} \times \text{vſ.}\, \beta$.

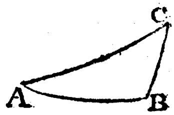
By the help of theſe theorems, knowing nearly what any quantity in a ſpherical triangle is, we may find its correction, thus: if we have to find the coſine of an arc, which arc we know is nearly equal to α whoſe coſine is a. Suppoſe the arc to be $\alpha-\beta$, and its coſine $a+c$. Then $a+c =$ coſ. $\overline{\alpha-\beta} = a + \mathrm{AB} - a \times \text{vſ.}\, \beta$. Therefore, $\mathrm{B} = \frac{c}{\mathrm{A}} + \frac{a}{\mathrm{A}} \times \text{verſ.}\, \beta$.

 The

The firſt term $\frac{c}{A}$ will always give a near approximation to the value of ſin. β, and β being found the correction, $\frac{a}{A} \times$ vſ. β, or cot. $\alpha \times$ vſ. β, may be found and added to it. Among the tables requiſite to be uſed with the Nautical Almanac, is table IV. for parallax, p. 19. which ſhews the value at ſight of ſuch quantities as vſ. $\beta \times$ cot. α, the arc β being found in the firſt column of the table, and α at the top. This table I have calculated only to arcs under 63′; but it would be found uſeful to have a table ready computed for all arcs under 5°.

PROBLEM I.

If the two legs, AB *and* BC, *of the ſpherical triangle* ABC *right-angled at* B, *are given, to find the hypotenuſe* AC, *the leg* BC, *being ſmall in compariſon of* AC.



Let AB$=\alpha$, BC$=\beta$, and ſuppoſe AC$=\alpha+\zeta$, α being a near approximation to AC, and ζ the ſmall arc to be added to AB to make it equal to AC; then coſ. AC=coſ. AB × coſ. BC; that is, according to our notation, $a - \text{AZ} - \overline{a \times \text{vſ. } \zeta} = a\,b$.

Whence $\text{Z} = \frac{a - ab}{\text{A}} - \overline{\frac{a}{\text{A}}} \times \text{vſ. } \zeta = \text{cot. } \alpha \times \text{vſ. } \beta - \text{cot. } \alpha \times \text{vſ. } \zeta$.

EXAMPLE.

Let AB be 75° 0′ and BC 20° 0′, and the computation will be as follows:

Cotangent AB		9.4280
Verſed ſine BC		8.7804
ζ nearly	55′ 33″ ſine	8.2084
Correction	−7	from tab. IV. Nautical Almanac.

Therefore ζ= 55 26 and AC=75° 55′ 26″.

By this problem, the diſtance of the Sun may be found from a planet whoſe latitude and difference of longitude are known.

PROBLEM II.

Having the hypotenuse AC *and one of the angles* A, *to find the base* AB.

Let AC$=\beta$, BAC$=\alpha$, and suppose AB$=\beta-\zeta$, then cos. A$=$ cot. AC$\times$tang. AB, or $a=\frac{b}{B}\times\frac{B}{b}-\frac{z}{b^2}+\frac{B\times \text{vs.}\zeta}{b^3}=1-\frac{z}{Bb}+\frac{B\times\text{vs.}\zeta}{b^2}$.

Whence $z=Bb\times\overline{1-a}+\frac{B}{b}\times\text{vs.}\zeta=\frac{1}{2}\text{ sin. }2\beta\times\text{vs.}\alpha+\text{tang.}\beta\times\text{vs.}\zeta$.

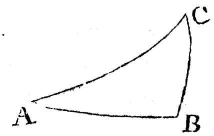
EXAMPLE.

Let A$=23°\ 28'\ 15''$, and AC$=10°\ 0'\ 0''$.

Sine 2 AC 20° 0′		9.5340
Versed sine A		8.9177
		8.4517
Log. 2.		0.3010
ζ nearly	48′ 39″ sine	8.1507
Correction	+ 6	
ζ	48 45 and BC$=159°\ 11'\ 15''$.	

By this problem, the right ascension of any point of the ecliptic, whose obliquity and longitude are known, may be found.

PROBLEM III.



Suppoſing the ſame things known as in the laſt, to find the perpendicular BC, *when the hypotenuſe is nearly a quadrant.*

Let A$=\alpha$, AC$=\beta$, as before, and ſuppoſe BC$=a-\zeta$; then ſin. BC = ſin. AC × ſin. A, or A$-a$Z$-$A × vſ. ζ = AB, whence $z = \frac{\text{A}-\text{BA}}{a} - \frac{\text{A}}{a} \times$ vſ. ζ = tang. $\alpha \times$ co. ver. ſin. $\beta -$ t.$\alpha \times$ vſ. ζ.

EXAMPLE.

Let A$=23^\circ\ 28'\ 15''$, and AC × $80^\circ\ 0'$.

Tang. A		9.6377
Verſ. ſin. co. AC, 10°		8.1816
ζ nearly	22° 41 ſine	7.8193
Correction	− 1	
ζ	22 40 and BC$=23^\circ\ 5'\ 35''$.	

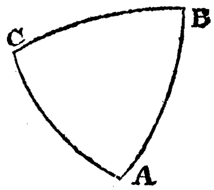
This problem will be of uſe to find the declination of the ecliptic, and the latitude of a planet near the limits.

Theſe three inſtances will ſuffice for an application of this method to right-angled ſpherical triangles; we ſhall now give two problems of oblique triangles.

PRO

PROBLEM IV.

Suppoſe ABC *to be a ſpherical triangle, in which are given the two ſides* AB, BC, *with the included angle* B, *to find the third ſide* AC.



SOLUTION I.

Let ABC $=\beta$, BC $=\alpha$, AB $=\delta$. Put AC $=\beta+\zeta$, β being an approximate value of AC, when the two legs are nearly quadrants. Now the coſine of AC being equal to $b\,\mathrm{DA}+da$[(a)] we ſhall have $b-\mathrm{BZ}-\overline{b\times\text{vſ.}\,\zeta}=b\,\mathrm{DA}+da$: and $\mathrm{Z}=\frac{b-b\,\mathrm{DA}-da}{\mathrm{B}}-\frac{b}{\mathrm{B}}\times\text{vſ.}\,\zeta$. But $1-\mathrm{DA}-da=\text{vſ.}\,\delta-\alpha$, which put $=w$. Then $\mathrm{Z}=\frac{bw}{\mathrm{B}}+\frac{b\,da-da}{\mathrm{B}}-\frac{b}{\mathrm{B}}\times\text{vſ.}\,\zeta=\text{cot.}\,\beta\times\text{vſ.}\,\overline{\delta-a}-\text{coſ.}\,\delta\times\text{coſ.}\,\alpha\times\text{tang.}\,\tfrac{1}{2}\beta-\text{cot.}\,\beta\times\text{vſ.}\,\zeta$. Therefore ζ is the difference of two arcs whoſe ſines are $\text{cot.}\,\beta\times\text{vſ.}\,\delta-\alpha$, and $\text{coſ.}\,\delta\times\text{coſ.}\,\alpha\times\text{tang.}\,\tfrac{1}{2}\beta$, the difference of theſe two arcs being diminiſhed by the correction $\text{cot.}\,\beta\times\text{vſ.}\,\zeta$.

(a) It is a well known theor. that $\text{ſin.}\,\mathrm{BA}\times\text{ſin.}\,\mathrm{BC}:r^2=\text{vſ.}\,\mathrm{AC}-\text{vſ.}\,\overline{\mathrm{AB}-\mathrm{BC}}:\text{vſ.}\,\mathrm{B}$; that is, $\text{ſin.}\,\mathrm{BA}\times\text{ſin.}\,\mathrm{BC}:r^2=\text{coſ.}\,\overline{\mathrm{AB}-\mathrm{BC}}-\text{coſ.}\,\mathrm{AC}:r-\text{coſ.}\,\mathrm{B}$. Or, in the author's notation, putting $r=1$, $\mathrm{DA}:1=\text{coſ.}\,\overline{\delta-\alpha}-\text{coſ.}\,\mathrm{AC}:1-b$. Therefore $\mathrm{DA}-b\,\mathrm{DA}=\text{coſ.}\,\overline{\delta-\alpha}-\text{coſ.}\,\mathrm{AC}$. Or, $\text{coſ.}\,\mathrm{AC}=b\,\mathrm{DA}-\mathrm{DA}-\text{coſ.}\,\overline{\delta-\alpha}$. For $\text{coſ.}\,\overline{\delta-\alpha}$ ſubſtitute its value as expreſſed in the ſecond corollary of the lemma, and there ariſes the author's equation, $\text{coſ.}\,\mathrm{AC}=b\,\mathrm{DA}+da$.

S. HORSLEY.

EXAMPLE.

EXAMPLE.

Suppose B = 51° 12′ 5″
AB = 87 57 51
BC = 87 20 34

Cotangent B	9.9053	Tang. ½ B 25° 36′	9.6804
Verf. fine AB—BC 0° 37′ 17″	5.7693	Cofine AB	8.5506
		Cofine BC	8.6661
1ft arc 0′ 10″ fine	5.6746	2d arc 2′ 43″ fine	6.8971

The difference of thefe two arcs,		2′	33″
Subtracted from the value of the angle B,	51	12	5
Leaves AC,	51	9	32

The correction cot. $\beta \times$ vf. ζ in this example is 0.

This folution is very convenient to find the diftance of two Zodiacal Stars, having their latitudes and difference of longitude.

SOLUTION II.

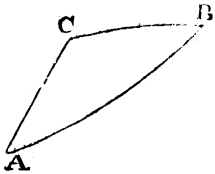
Let τ be an arc whofe cofine $t = b \times \text{cof.}\ \delta - \alpha = bda + b\text{DA}$, and fuppofe AC $= \tau - \zeta$, then $t + \text{T}z - t \times \text{vf.}\ \zeta = b\text{DA} + da$ $= t - bda + da$. Whence $z = da \times \frac{1-b}{\text{T}} + \frac{t}{\text{T}} \times \text{verf.}\ \zeta =$ cofec. τ $\times$ cofin. $\alpha \times$ cofin. $\delta \times$ vf. $\beta +$ cot. $\tau \times$ vf. ζ.

This folution is ufeful to find the diftance of the Moon from a ftar at fome diftance from the ecliptic, in which cafe it coincides with the rule given by the Aftronomer

Royal,

Royal, Phil. Tranſ. 1764, vol. LIV. and which taking in the correction here given cot. $\tau \times$ vſ. ζ will always be exact to a ſecond. It is alſo of uſe to find the declination of a ſtar, whoſe longitude and latitude and obliquity of the ecliptic are known.

SOLUTION III.



Let the angle B be ſmall, and the two legs AB, BC, very unequal; then the ſide AC will be nearly AB − BC. Put this $= x$, and ſuppoſe AC $= x + \zeta$, then coſ. AC $= k - \mathrm{KZ} - k \times$ vſ. $\zeta = ad + \mathrm{AD} - \mathrm{KZ} - k \times$ vſ. $\zeta = b\,\mathrm{AD} + ad$, Whence $\mathrm{Z} = \frac{\mathrm{DA} - b\,\mathrm{AD}}{\mathrm{K}} - \frac{k}{\mathrm{K}} \times$ vſ. $\zeta =$ ſin. $\delta \times$ ſin. $\alpha \times$ vſ. $\beta \times$ coſec. $\delta - \alpha -$ cot. $k \times$ vſ. ζ.

EXAMPLE.

Let AB $= 94^\circ\ 36'\ 58''$, BC $= 23\ \ 28\ \ 24$, B $= 24\ \ 54\ \ 24$ — as in the example to ſol. 2.

AB − BC $= 71\ \ 8\ \ 34$	Coſecant	0.02396
Sine AB		9.99859
Sine BC		9.60023
Verſed of B		8.96851
ζ nearly $2^\circ\ 14'\ 11''$ ſine		8.59129

The value of ζ being without the limits of tab. IV. in the tables requisite to be used with the Nautical Almanac, the correction cot. $\varkappa \times$ vs. ζ must be computed thus:

Cot. $\varkappa$	9.533
V. sin. ζ	6.881

Cor. 0′ 53″. sine 6.414, this subtracted from the first value of ζ, leaves $\zeta = 2^\circ\ 13'\ 18''$, which added to $\delta - \alpha$, gives the side AC$=73^\circ\ 21'\ 52''$. This solution will help to find the Sun's altitude near noon.

I have dwelt the longer on this problem because it is one that is very commonly required in astronomical calculations, and the operation by the rules of spherical trigonometry in this as well as the next is rather troublesome.

PROBLEM V.

Suppoſing the ſame things given, to find either of the angles, as for inſtance C *oppoſite the ſide* AB.

We have cot.C=cot. B × coſ.BC − ſin.BC × cot. AB × coſec. B $= \frac{ba}{B} - \frac{Ad}{BD}$. Let μ be an angle whoſe cot. $\frac{m}{M}$ = cot. β × ſin. $\delta - \alpha$ × coſec. $\delta = \frac{ba D - bAd}{BD}$, and ſuppoſe $C = \mu + \zeta$, then cot $C = \frac{m}{M} - \frac{z}{M^2} + \frac{m.\text{vſ.}\zeta}{M^3} = \frac{ba D - Ad}{BD}$. Whence $z = M \times \frac{Ad - bAD}{BD} + \frac{m}{M} \times \text{vſ.}\ \zeta = \overline{\text{ſin.}\ \mu}^2 \times \text{ſin.}\ \alpha \times \text{cot.}\ \delta \times \text{tang.}\ \frac{1}{2}\beta + \text{cot.}\ \mu \times \text{vſ.}\ \zeta$.

EXAMPLE.

Let	AB=94°	36′	58.		
	BC=23	28	54		
	B=24	54	24	Cotang.	0.3331770
Diff. AB and BC=71		8	34	Sine	9.9760412
				Coſecant AB	0.0014080
	μ= 26	3	44	Cot.	10.3166262

$\overline{\text{Sin.}\ \mu}^2$	9.286
Sin. BB	9.600
Cot. AB	8.909
Tang. $\frac{1}{2}$B	9.344

$\zeta=4'\ 44''$ ſine 7.139, this ſubtracted from μ leaves the angle C=25° 59′ 0″

This

This problem will be of uſe to find the right aſcenſion of a ſtar whoſe longitude and latitude, and obliquity of the ecliptic are known, or to find the Sun's azimuth at any hour in a given latitude.

I have added no cautions when theſe approximations and corrections change their ſigns, becauſe any mathematician will diſcover them at ſight.

I have the honour to be, &c.